Physical examples of line integrals:

- The total work done by a force $\mathbf{F}$ which moves its application point along a given curve $C$ is given by

$$W_C = \int_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{r}$ is the position vector of the application point at a given moment.

- If a loop of wire $C$ carrying a current $I$ is placed in a magnetic field $\mathbf{B}$ then the total force on the wire is

$$\mathbf{F} = I \oint_C d\mathbf{r} \times \mathbf{B}$$
If \( \mathbf{A} = A_0 \mathbf{i} + A_2 \mathbf{j} + A_3 \mathbf{k} \) is a vector then its integral along a curve \( C \), from a point \( P_1 \) to another point \( P_2 \) can be calculated as

\[
\int_{P_1}^{P_2} \mathbf{A} \cdot d\mathbf{r} = \int_C \mathbf{A} \cdot d\mathbf{r} = \int_C (A_1 \mathbf{i} + A_2 \mathbf{j} + A_3 \mathbf{k}) \cdot (dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}) = \int_C A_1 dx + A_2 dy + A_3 dz
\]

which can then be calculated in the usual manner once the equation of the curve \( C \) is known.
Example

Evaluate the line integral $\int_C \mathbf{A} \cdot d\mathbf{r}$ where $\mathbf{A} = (x + y)\mathbf{i} + (y - x)\mathbf{j}$ along each of the following 2-dimensional curves

1. the parabola $y^2 = x$ from $(1,1)$ to $(4,2)$;
2. the curve $x = 2t^2 + t + 1$, $y = 1 + t^2$ from $(1,1)$ to $(4,2)$;
3. the line $y = 1$ from $(1,1)$ to $(4,1)$ followed by the line $x = 4$ from $(4,1)$ to $(4,2)$.

Solution: Write the integral as

$$\int_C \mathbf{A} \cdot d\mathbf{r} = \int_C (x + y)\,dx + (y - x)\,dy$$
1. Along the parabola \( x = y^2 \) we have \( dx = 2ydy \). Substitute all \( x \) in the integral in terms of \( y \) to get

\[
I = \int_1^2 [(y^2 + y)2y + (y - y^2)] \, dy = \frac{34}{3}
\]

2. We write \( x \) and \( y \) in terms of \( t \) so \( dx = (4t + 1) \, dt \) and \( dy = 2t \, dt \). The limits for \( t \) can be evaluated from the limits for \( x \) and \( y \).

\[
I = \int_0^1 [(3t^2 + t = 2)(4t + 1) - (t^2 + t)2t] \, dt = \frac{32}{3}
\]

3. The line integral must be evaluated along the two line segments separately and the results added together. Note that along the line \( y = 1 \) we have \( dy = 0 \) while along the line \( x = 4 \) we have \( dx = 0 \). So

\[
I = \int_1^4 (x + 1) \, dx + \int_1^2 (y - 4) \, dy = -\frac{5}{2}.
\]
More examples

Example 2: Evaluate the line integral

\[ I = \oint_C x \, dy \]

where \( C \) is the circle in the \( xy \)-plane defined by \( x^2 + y^2 = 1 \).

Example 3: If

\[ \mathbf{A} = (3x^2 - 6yz)\mathbf{i} + (2y + 3xz)\mathbf{j} + (1 - 4xyz^2)\mathbf{k} \]

evaluate \( \int_C \mathbf{A} \cdot d\mathbf{r} \) from \((0,0,0)\) to \((1,1,1)\) along the following paths \( C \):

1. \( x = t, \ y = t^2, \ z = t^3; \)
2. the straight lines from \((0,0,0)\) to \((0,0,1)\), then to \((0,1,1)\) and then to \((1,1,1)\);
3. the straight line joining \((0,0,0)\) to \((1,1,1)\)
Green’s theorem in the plane

Green’s theorem, also called the **divergence theorem in two dimensions** relates a line integral around a closed curve, $C$, to a double integral over the region $R$ enclosed by the curve,

$$\oint_C (P \, dx + Q \, dy) = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dxdy.$$

Green’s theorem gives a condition for a line integral to be independent of its path.
Independence of the path

Consider the line integral

\[ I = \int_{A}^{B} (P \, dx + Q \, dy) \]

We say that this integral is independent of the path taken from \( A \) to \( B \) if it has the same value along any two arbitrary paths \( C_1 \) and \( C_2 \) between the points.

This means that, if we take \( C = C_1 - C_2 \) (the closed loop formed by \( C_1 \) and \( C_2 \), the integral around \( C \) must be zero.

It can be seen, from Green’s theorem, that a necessary condition for the line integral to be zero is

\[ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \]

This is also a sufficient condition for path independence.
Examples

1. Show that the area bounded by a closed curve $C$ is given by

$$\frac{1}{2} \oint_C xdy - ydx$$

Hence calculate the area of the ellipse, $x = a\cos(t), y = b\sin(t)$.

2. Show that the integral

$$\int_{(1,2)}^{(3,4)} \left( 6xy^2 - y^3 \right) dx + (6x^2 y - 3xy^2) dy$$

is independent of the path joining the two points. Evaluate the integral.
Conservative vector fields

A vector field $\mathbf{A}$ is called **conservative** if any of the following equivalent conditions holds

- The line integral of $\mathbf{A}$ between two points is independent of the path
- The line integral of $\mathbf{A}$ over any closed curve $C$ is equal to zero, that is
  $$\oint_C \mathbf{A} \cdot d\mathbf{r} = 0$$
- The curl of $\mathbf{A}$ is zero, $\nabla \times \mathbf{A} = 0$;
- There exists a scalar field $\Phi(x, y, z)$, called a **potential**, such that
  $$\mathbf{A} = \nabla \Phi$$

**Example:** Show that $\mathbf{A} = (xy^2 + z)\mathbf{i} + (x^2y + 2)\mathbf{j} + x\mathbf{k}$ is conservative and find its scalar potential $\Phi$. 
The divergence theorem

Let $S$ be a closed surface bounding a region of volume $V$ and let $\mathbf{n}$ be the unit (outward) normal to the surface.

The divergence theorem states that

$$\iiint_S \mathbf{A} \cdot \mathbf{n} \, dS = \iiint_V \nabla \cdot \mathbf{A} \, dV$$

In other words, the surface integral of the normal component of a vector $\mathbf{A}$ taken over a closed surface is equal to the integral of the divergence of $\mathbf{A}$ taken over the volume enclosed by the surface.
Calculating surface integrals

**Note:** The surface integral $\iint_S A \cdot n \, dS$ is sometimes written as

$$\iint_S A \cdot dS$$

where $dS$ and $dS$ are referred to as the scalar and vector area elements, respectively, with $dS = n \, dS$.

If the surface $S$ is given by the equation $z = F(x, y)$ then a surface integral over $S$ is calculated as follows

$$\iint_S \Phi \, dS = \iint_R \Phi \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} \, dx \, dy$$

where $R$ is the projection of $S$ onto the $xy$ plane. Similar formulas hold for projections onto the $xz$ or $yz$ planes.
Evaluate
\[ \iiint_S A \cdot n \, dS \]
where \( A = xy\mathbf{i} - x^2\mathbf{j} + (x + z)\mathbf{k} \), \( S \) is that portion of the plane \( 2x + 2y + z = 6 \) included in the first octant and \( n \) is a unit normal to \( S \).
Example 2

Verify the divergence theorem for \( \mathbf{A} = (2x - z)i + x^2yj - xz^2k \) taken over the region bounded by \( x = 0, \ x = 1, \ y = 0, \ y = 1, \ z = 0, \ z = 1. \)

To verify the divergence theorem, calculate first the volume integral:

\[
\iiint_V \nabla \cdot \mathbf{A} \, dV = \int_0^1 \int_0^1 \int_0^1 (2 + x^2 - 2xz) \, dx \, dy \, dz = \frac{11}{6}
\]
Next evaluate $\iint_S \mathbf{A} \cdot \mathbf{n}$ on each face of the cube.

1. On face $\text{AFEB}$ we have $\mathbf{n} = \mathbf{i}$ and $x = 1$. Then

$$\iint_{\text{AFEB}} \mathbf{A} \cdot \mathbf{n} \, dS = \int_0^1 \int_0^1 [(2 - z)\mathbf{i} + \mathbf{j} - z^2 \mathbf{k}] \cdot \mathbf{i} \, dydz = \frac{3}{2}$$

2. On face $\text{COGD}$ we have $\mathbf{n} = -\mathbf{i}$ and $x = 0$.

$$\iint_{\text{COGD}} \mathbf{A} \cdot \mathbf{n} \, dS = \int_0^1 \int_0^1 (-z\mathbf{i}) \cdot (-\mathbf{i}) \, dydz = \frac{1}{2}$$

3. On face $\text{BEDC}$ we have $\mathbf{n} = \mathbf{j}$ and $y = 1$.

$$\iint_{\text{BEDC}} \mathbf{A} \cdot \mathbf{n} \, dS = \int_0^1 \int_0^1 [(2x - z)\mathbf{i} + x^2 \mathbf{j} - xz^2 \mathbf{k}] \cdot \mathbf{j} \, dxdz = \frac{1}{3}$$
4. On face OAFG we have $\mathbf{n} = -\mathbf{j}$ and $y = 0$

$$\iint_{OAFG} \mathbf{A} \cdot \mathbf{n} \, dS = \int_0^1 \int_0^1 [(2x - z)\mathbf{i} - xz^2\mathbf{k}] \cdot (-\mathbf{j}) \, dx \, dz = 0$$

5. On face EFGD we have $\mathbf{n} = \mathbf{k}$ and $z = 1$

$$\iint_{EFGD} \mathbf{A} \cdot \mathbf{n} \, dS = \int_0^1 \int_0^1 [(2x - 1)\mathbf{i} + x^2y\mathbf{j} - x\mathbf{k}] \cdot \mathbf{k} \, dx \, dy = -\frac{1}{2}$$

6. On OABC we have $\mathbf{n} = -\mathbf{k}$ and $z = 0$

$$\iint_{OABC} \mathbf{A} \cdot \mathbf{n} \, dS = \int_0^1 \int_0^1 [2x\mathbf{i} - x^2y\mathbf{j}] \cdot (-\mathbf{k}) \, dx \, dy = 0$$

Adding the six faces we get $\frac{3}{2} + \frac{1}{2} + \frac{1}{3} + 0 - \frac{1}{2} + 0 = \frac{11}{6}$. 
Stokes’ Theorem

The line integral of a vector $\mathbf{A}$ taken around a simple closed curve (that is, a non-intersecting closed curve), $C$, is equal to the surface integral of the curl of $\mathbf{A}$ taken over any surface $S$ having $C$ as a boundary.

$$\int_C \mathbf{A} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

Note that, if $\nabla \times \mathbf{A} = 0$ then the line integral of $\mathbf{A}$ over the closed curve $C$ is zero and hence the vector field is conservative.
Physical examples: Surface Integrals

If $A$ is a vector field, the surface integral

$$\iint_S A \cdot dS$$

is called the flux of $A$ through the surface $S$.

**Gauss’ flux theorem:** The electric flux through any closed surface is proportional to the enclosed electric charge

$$Q = \varepsilon_0 \iiint_S E \cdot dS$$

where $\varepsilon_0$ is the permittivity of free space or electric constant.
Physical examples: Volume integrals

The volume of a closed region in space, \( D \), is given by

\[
V = \iiint_D dV
\]

The mass of an object which occupies a region \( D \) and has density \( \delta(x, y, z) \) is given by

\[
M = \iiint_D \delta \, dV
\]

Total electric charge enclosed within a region \( D \) is

\[
Q = \iiint_V \rho \, dV
\]

where \( \rho(x, y, z) \) is the charge density.
Applications of the divergence theorem

The integral form of the Gauss equation is

\[ Q = \varepsilon_0 \oiint_S \mathbf{E} \cdot d\mathbf{S} \]

which can be written as

\[ \iiint_V \rho \, dV = \varepsilon_0 \iiint_V \nabla \cdot \mathbf{E} \, dV \]

and hence

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \]

which is the differential form.
Applications of the Stokes theorem

The Stokes’ Theorem shows the equivalence between the integral and differential forms of the Ampere-Maxwell equation.

Ampere’s law for a distributed current $I$ with current density $J$ is

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I = \mu_0 \int_S J \cdot d\mathbf{S}$$

for any circuit $C$ bounding a surface $S$. Using Stokes’ Theorem, this can be written as

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \mu_0 \int_S J \cdot d\mathbf{S}$$

and, since this holds for any surface $S$, it follows that

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$